H-bases and lifting problem for homogeneous ideals

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Abstract

The lifting problem for homogeneous ideals in a polynomial ring over a field is studied. A division algorithm for H-bases is given. Using this division algorithm, a new method for finding the liftings of a homogeneous ideal is developed. This method was compared with the current methods. The results are demonstrated with examples.

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1 Introduction

Let $A = K[x_1, x_2, \ldots, x_{n-1}]$ and $B = K[x_1, x_2, \ldots, x_{n-1}, x_n]$ where K is a field.

Definition 1.1. Let J be a homogeneous ideal in the polynomial ring A. A homogeneous ideal I in B is called lifting of J if

(i) x_n is not a zero divisor on B/I; and

(ii) $J = \langle f(x_1, x_2, \ldots, x_{n-1}, 0) : f \in I\rangle$.

The problem of finding the liftings of a given homogeneous ideal first suggested in [1] and fully explained in [2]. Then this interesting problem has been investigated by many researchers (see [3, 4, 5, 6]).

More recently a new computational method for finding the liftings of a given homogeneous ideal is given [7]. This method involves some Gröbner basis computations. Here we will give only a few definitions that are sufficient to explain the method given in [7]. We refer to [8] for a detailed treatment of Gröbner bases.

For a given a monomial order $<$ and a polynomial f in a polynomial ring, $\mathrm{LT}(f)$ denotes the leading term of f with respect to $<$. If J is an ideal in this polynomial ring $\mathrm{LT}(J) = \langle \mathrm{LT}(f) : f \in J\rangle$ and $\mathcal{N}_J$ is the set of monomials which are not in $\mathrm{LT}(J)$.

Now we are ready to explain the method given in [7]. Starting with a homogeneous ideal $J = \langle f_1, \ldots, f_s\rangle \subset A$ compute a Gröbner basis $\{h_1, \ldots, h_t\}$ of J with respect to a given monomial order $<$ in A. After that define the polynomials

$$g_i = h_i + \sum_{\substack{x^\alpha x_n \in \mathcal{N}_J \\ \deg(x^\alpha x_n)=\deg(h_i)}} C_{i\alpha} x^\alpha x_n$$

where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$.

The set of polynomials $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis for the ideal $I = \langle G\rangle$ with respect to a special extension of monomial order$<$ into a monomial order in B if and only if I is a lifting

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of J. Buchberger's criterion for Gröbner bases implies the remainders of S-polynomials $S(g_i, g_j)$ on division by G is zero for $1 \leq i < j \leq t$. Hence equalizing the coefficients of remainders to zero for each division gives us the conditions that the parameters must satisfy for I to be a lifting of J. Theoretical background of this method depends on a reformulation of a theorem given [4] in terms of Gröbner bases. In the original theorem however the lifting problem is related to the H-bases not to the Gröbner bases. Therefore, this method is against to the nature of the problem. It finds a Gröbner basis of a lifting which is not essential. Because of this, the method involves many unnecessary computations.

The first attempt to use H-bases to solve the lifting problem was given in [6]. The authors gave a criterion for H-bases in terms of syzygy modules which is very similar to Buchbergers criterion for Gröbner bases. Because they could not use the division algorithm on H-bases, they were not able to give a complete method for solving the lifting problem.

In this paper, we use the vector space character of homogeneous ideals to define a kind of division on homogeneous polynomials. Combining the ideas given in [6] with this division algorithm, we developed a new method for solving the lifting problem for homogeneous ideals. This method is similar to the method given in [7] but it uses H-bases instead of Gröbner bases. Finally, we compare these two methods and explain the advantages of our method.

2 H-bases

If $f = f_d + f_{d-1} + \cdots + f_1 + f_0$ where each $f_i \in B$ is a homogeneous polynomial of degree i and $f_d \neq 0$, then $H(f) = f_d$ is called the leading form of f. For an ideal I of A, $H(I) = \langle H(f) : f \in I \rangle$.

Definition 2.1. A set of polynomials $\{f_1, f_2, \ldots, f_s\} \subset B$ is an H-basis for the ideal $I = \langle f_1, f_2, \ldots, f_s \rangle$ provided that $H(I) = \langle H(f_1), H(f_2), \ldots, H(f_s) \rangle$.

Let $f = f_d + f_{d-1} + \cdots + f_1 + f_0$ be a polynomial in the polynomial ring A. The homogenization of f with respect to x_n is defined to be the polynomial $f^h = f_d + x_n f_{d-1} + \cdots + x_n^{d-1} f_1 + x_n^d f_0 \in B$. For an ideal I of A the homogenization of I, denoted by I^h, is the ideal $I^h = \langle f^h : f \in I \rangle \subseteq B$.

Let f be a homogeneous polynomial in B, then the dehomogenization of f is the polynomial $f_a = f(x_1, x_2, \ldots, x_{n-1}, 1) \in A$. For a homogeneous ideal I of B, the dehomogenization of I is the ideal $I_a = \langle f_a : f \in I \rangle \subseteq A$.

Lemma 2.2. [4, Lemma 2.4] Let I be a homogeneous ideal of B. Then the following conditions are equivalent.

(i) x_n is not a zero divisor on B/I,

(ii) $I = (I_a)^h$,

(iii) $H(I_a) = I(x_1, x_2, \ldots, x_{n-1}, 0)$.

The next theorem gives a relation between the lifting problem and H-bases. A Gröbner basis version of this theorem is used in [7].

Theorem 2.3. [4, Theorem 2.5] Let $J = \langle f_1, f_2, \ldots, f_s \rangle$ be a homogeneous ideal of A.

(i) Let $g_i = f_i + R_i$ with $deg(R_i) < deg(f_i)$ for $1 \leq i \leq s$ and $I = \langle g_1, g_2, \ldots, g_s \rangle \subseteq A$. If $\{g_1, g_2, \ldots, g_s\}$ is an H-basis for I, then $I^h = \langle g_1^h, g_2^h, \ldots, g_s^h \rangle$ is a lifting of J. Conversely,

(ii) If I is a lifting of J, then there exist polynomials $R_1, R_2, \ldots, R_s \in A$ such that $deg(R_i) < deg(f_i)$ for every i, $\{g_1, g_2, \ldots, g_s\}$ is an H-basis and $I = \langle g_1^h, g_2^h, \ldots, g_s^h \rangle$.

This theorem is used in original form to find the liftings of a homogeneous ideal in[6]. We try to explain their method but we need to define the syzygy module of a set of polynomials.

Definition 2.4. For an s-tuple of polynomials $(f_1, \ldots, f_s)$, the module generated by following set of s-tuple of polynomials

$$\{(h_1, \ldots, h_s) : h_1 f_1 + \cdots + h_s f_s = 0\}$$

is called syzygy module of $(f_1, \ldots, f_s)$ and denoted by $syz(f_1, \ldots, f_s)$.

The following theorem gives a criterion for a set of polynomials to be an H-basis. This criterion is H-basis version of Buchberger's criterion for a Gröbner basis.

Theorem 2.5. [6, Theorem 2.4] Let $I = \langle h_1, h_2, \ldots, h_t \rangle \subseteq K[x_1, \ldots, x_n]$. Let the columns of the $t \times l$ matrix $S = (s_{ij})$ be a generating set of $syz(H(h_1), \ldots, H(h_t))$. We may assume further that each $s_{ji} f_j$ is a homogeneous polynomial of same degree for $j = 1, \ldots, t$. Then $\mathcal{H} = \{h_1, h_2, \ldots, h_t\}$ is an H-basis for I if and only if

$$q_i = \sum_{j=1}^{t} s_{ji} h_j = \sum_{j=1}^{t} a_{ji} h_j, \quad 1 \leq i \leq l$$

for some $a_{ji} \in K[x_1, \ldots, x_n]$ such that $deg(q_i) = \max\{deg(a_{ji} h_j), j = 1, \ldots, t\}$.

The following method for finding the liftings of a homogeneous ideal is proposed in [6]: Given an ideal $J = \langle f_1, \ldots, f_r \rangle \subseteq K[x_1, \ldots, x_n]$ where f_i's are homogeneous, define

$$g_i = \sum_{deg(x^\gamma) < deg(f_i)} C_{i\gamma} x^\gamma, \quad h_i = f_i + g_i.$$

Furthermore, for each q_i in Theorem 2.5 define

$$a_{ji} = \sum_{deg(x^\gamma) < deg(q_i) - deg(h_j)} D_{i\gamma} x^\gamma.$$

Then compare the coefficient of monomials of the equation given Theorem 2.5 to find relations among the parameters $C_{i\gamma}$'s and $D_{ij\gamma}$'s. Then $I = \langle h_1, h_2, \ldots, h_r \rangle$ is an element of the liftings of J if and only if the coefficients of h_i's satisfy these relations. This is not a convenient method because there are extra parameters $D_{ij\gamma}$'s. Even though in their example they are able to solve these extra parameters in terms of $C_{i\gamma}$'s, there is no guarantee that this will always occur.

The effectiveness of the method proposed in [7] comes from the usage of the division algorithm. This is the weak part of the method in [6]. Because of this we define a division process for H-bases.

3 A division algorithm for H-bases

Let P_d be the vector space of homogeneous polynomials of degree d in variables $x_1, x_2, \ldots, x_n$ over a field K. For brevity we denote the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n}$ by x^α. It is well-known that

$$\mathcal{B} = \{x^\alpha : \deg(x^\alpha) = \alpha_1 + \alpha_2 + \ldots + \alpha_n = d\}$$

is a basis for P_d.

Let $I = \langle f_1, f_2, \ldots, f_s \rangle \subseteq K[x_1, x_2, \ldots, x_n]$ where f_i's are homogeneous polynomials. $V_d(I) = \{f \in I : deg(f) = d \quad or \quad f = 0\}$ is a subspace of P_d. Furthermore, the polynomials of the form $x_1^{\alpha_1} \ldots x_n^{\alpha_n} f_i$ where $\deg(x^\alpha) + \deg(f_i) = d$ spans $V_d(I)$.

For a given homogeneous polynomial $f \in K[x_1, x_2, \ldots, x_n]$ of degree d, $f \in I$ if and only if $f \in V_d(I)$. We can decide whether $f \in V_d(I)$ or not as follows:

Let $f = \sum_{\deg(x^\alpha)=d} a_i x^\alpha$ where $a_i \in K$. We have to find c_{β_j}'s in K such that

$$f = \sum_{\deg(x^\beta)+\deg(f_j)=d} c_{\beta_j} x^\beta f_j.$$

This is just a system of linear equations and can be solved linear algebra techniques. Construct the matrix M whose columns are coordinate vectors of $x^\beta f_j$'s with respect to $\mathcal{B}$. Also the last column of M is the coordinate vector of f with respect to $\mathcal{B}$. If $M^{'}$ is the row reduced echelon form of M, then these matrices will give the solution of same system since they are row equivalent. If $M^{'}$ has any row like $(0, 0, \ldots, 0, L(a_1, a_2, \ldots))$ where L is a linear function, then the system has no solution unless $L(a_1, a_2, \ldots) = 0$. Such rows will give relations between a_i's for f to be in $V_d(I)$.

Example 3.1. Consider the homogeneous ideal

$$I = \langle f_1, f_2, f_3 \rangle = \langle x_1^2 + x_1x_3, x_1x_2 + x_2x_3, x_1^3 + x_1x_2^2 + x_2^2x_3 \rangle$$

and the polynomial

$$g = a_1x_1^3 + a_2x_1^2x_2 + a_4x_1x_2^2 + a_7x_2^3 + a_3x_1^2x_3 + a_5x_1x_2x_3 + a_8x_2^2x_3 + a_6x_1x_3^2 + a_9x_2x_3^2 + a_{10}x_3^3$$

of degree 3. We try to obtain the equations that a_i's must satisfy for the polynomial g to be in $V_3(I)$.

Clearly, $\{x_1f_1, x_2f_1, x_3f_1, x_1f_2, x_2f_2, x_3f_2, f_3\}$ is a spanning set for the vector space $V_3(I)$. Consider the augmented matrix M,

	x_1f_1	x_2f_1	x_3f_1	x_1f_2	x_2f_2	x_3f_2	f_3	g
x_1^3	1	0	0	0	0	0	1	a_1
$x_1^2x_2$	0	1	0	1	0	0	0	a_2
$x_1^2x_3$	1	0	1	0	0	0	0	a_3
$x_1x_2^2$	0	0	0	0	1	0	1	a_4
$x_1x_2x_3$	0	1	0	1	0	1	0	a_5
$x_1x_3^2$	0	0	1	0	0	0	0	a_6
x_2^3	0	0	0	0	0	0	0	a_7
$x_2^2x_3$	0	0	0	0	1	0	1	a_8
$x_2x_3^2$	0	0	0	0	0	1	0	a_9
x_2^3	0	0	0	0	0	0	0	a_{10}

After applying a series of elementary row operations, the row reduced echelon form of M, say $M^{'}$, will be

	x_1f_1	x_2f_1	x_3f_1	x_1f_2	x_2f_2	x_3f_2	f_3	g
x_1^3	1	0	0	0	0	0	0	$-a_6+a_3$
$x_1^2x_2$	0	1	0	1	0	0	0	a_2
$x_1^2x_3$	0	0	1	0	0	0	0	a_6
$x_1x_2^2$	0	0	0	0	1	0	0	$a_4-a_6+a_3-a_1$
$x_1x_2x_3$	0	0	0	0	0	1	0	a_5-a_2
$x_1x_3^2$	0	0	0	0	0	0	1	$a_6-a_3+a_1$
x_2^3	0	0	0	0	0	0	0	a_7
$x_2^2x_3$	0	0	0	0	0	0	0	a_8-a_4
$x_2x_3^2$	0	0	0	0	0	0	0	$a_9-a_5+a_2$
x_2^3	0	0	0	0	0	0	0	a_{10}

Hence, from the last 4 rows of $M^{'}$, $g \in V_3(I)$ if and only if

$$\begin{aligned} a_7 &= 0 \\ a_4 &= a_8 \\ a_9 &= a_5-a_2 \\ a_{10} &= 0 \end{aligned}$$

Under these conditions,
$g = [(-a_6+a_3)x_1 + a_2x_2 + a_6x_3]f_1 + [(a_4-a_6+a_3-a_1)x_2 + (a_5-a_2)x_3]f_2 + (a_6-a_3+a_1)f_3.$

Now we are ready to define a new division process.

Definition 3.2. Let $\mathcal{H} = \{f_1, f_2, \ldots, f_s\} \subseteq K[x_1, \ldots, x_n]$. For $f \in K[x_1, \ldots, x_n]$ we say f reduces to $\widetilde{f}$ modulo $\mathcal{H}$, written
$f \longrightarrow_{\mathcal{H}} \widetilde{f}$, if

$$\widetilde{f} = f - (a_1f_1 + \cdots + a_sf_s)$$

for some homogeneous polynomials $a_1, \ldots, a_s$ satisfying

$$H(f) = a_1H(f_1) + \cdots + a_sH(f_s)$$

and $deg(a_i) = deg(f) - deg(f_i)$.

We say f completely reduce to r, written $f \longrightarrow_{\mathcal{H}}^{+} r$, if there exists a sequence of polynomials $g_1, \ldots, g_t$ such that

$$f \rightarrow_{\mathcal{H}} g_1 \rightarrow_{\mathcal{H}} g_2 \rightarrow_{\mathcal{H}} \cdots \rightarrow_{\mathcal{H}} g_t \rightarrow_{\mathcal{H}} r$$

and no homogeneous part of r is in $\langle H(f_1), \ldots, H(f_s) \rangle$.

Lemma 3.3. Let $\mathcal{H} = \{f_1, f_2, \ldots, f_s\} \subseteq K[x_1, \ldots, x_n]$. For every $f \in K[x_1, \ldots, x_n]$, there exists $r \in K[x_1, \ldots, x_n]$ such that $f \longrightarrow_{\mathcal{H}}^{+} r$.

Proof. The desired polynomial r can be found using the following algorithm.

$h := f, r := 0$

WHILE $h \neq 0$ DO

IF $H(h) \in \langle H(f_1), \ldots, H(f_s) \rangle$ THEN

$h := \widetilde{h}$ where $h \longrightarrow_{\mathcal{H}} \widetilde{h}$

ELSE

$h := h - H(h)$

$r := r + H(h)$

Q.E.D.

Let us illustrate the algorithm with an example.

Example 3.4. Let $\mathcal{H} = \{f_1, f_2\} = \{x_1^3 + x_2x_3, x_1x_2 + x_3\}$ and $f = x_1^3 + x_1x_2^2 + x_2x_3 + x_1^2 + x_3$. We will apply the algorithm to find the polynomial r.

- Here $h = f$ and $r = 0$. Using technique given in Example 3.1,

$$H(h) = x_1^3 + x_1x_2^2 = H(f_1) + x_2H(f_2).$$

So, $h = \widetilde{h} = h - f_1 - x_2f_2 = x_1^2 - x_2x_3 + x_3$.

- Since $H(h) = x_1^2 - x_2x_3 \notin \langle H(f_1), H(f_2) \rangle$,

$$h = h - H(h) = x_3$$

and

$$r = 0 + H(h) = x_1^2 - x_2x_3.$$

- Now, $h \neq 0$ and $H(h) = x_3 \notin \langle H(f_1), H(f_2) \rangle$. So,

$$h = h - H(h) = 0$$

and

$$r = r + H(h) = x_1^2 - x_2x_3 + x_3.$$

Since $h = 0$, the algorithm ends with $r = x_1^2 - x_2x_3 + x_3$. Therefore, $f \longrightarrow_{\mathcal{H}}^{+} x_1^2 - x_2x_3 + x_3$.

4 New method

The following results give some relations between H-bases and the division algorithm defined on the previous section.

Lemma 4.1. Let $\mathcal{H} = \{f_1, f_2, \ldots, f_s\} \subseteq K[x_1, \ldots, x_n]$ and $I = \langle \mathcal{H} \rangle$. $\mathcal{H}$ is an H-basis for I if and only if for every $f \in I$, $f \longrightarrow^+_{\mathcal{H}} 0$.

Proof. Suppose that $\mathcal{H} = \{f_1, f_2, \ldots, f_s\}$ is an H-basis for $I = \langle \mathcal{H} \rangle$ and $f \in I$. Then there exist homogeneous polynomials $a_1, \ldots, a_s \in K[x_1, \ldots, x_n]$ satisfying $deg(a_i) = deg(f) - deg(f_i)$ such that $H(f) = a_1 H(f_1) + \cdots + a_s H(f_s)$. Notice that these polynomials can be obtained by the simple linear algebra techniques. Now define $\widetilde{f} = f - (a_1 f_1 + \cdots + a_s f_s)$. So, $f \longrightarrow_{\mathcal{H}} \widetilde{f}$. It is clear that $deg(\widetilde{f}) < deg(f)$ and $\widetilde{f} \in I$. Hence we can apply same process to the polynomial $\widetilde{f}$ and continue doing this until we reach to zero polynomial.

Conversely, suppose that $f \in I$ and $f \longrightarrow^+_{\mathcal{H}} 0$. Then there exists a sequence of polynomials $g_1, \ldots, g_t$ such that

$$f \rightarrow_{\mathcal{H}} g_1 \rightarrow_{\mathcal{H}} g_2 \rightarrow_{\mathcal{H}} \cdots \rightarrow_{\mathcal{H}} g_t \rightarrow_{\mathcal{H}} 0.$$

Since $f \rightarrow_{\mathcal{H}} g_1$, there exist homogeneous polynomials $b_1, \ldots, b_s$ such that $H(f) = b_1 H(f_1) + \cdots + b_s H(f_s)$ which implies that $H(f) \in \langle H(f_1), \ldots, H(f_s) \rangle$. Hence $H(I) = \langle H(f_1), \ldots, H(f_s) \rangle$, that means $\mathcal{H} = \{f_1, f_2, \ldots, f_s\}$ is an H-basis.

Q.E.D.

Lemma 4.2. Let $\mathcal{H} = \{f_1, f_2, \ldots, f_s\} \subseteq K[x_1, \ldots, x_n]$ and $I = \langle \mathcal{H} \rangle$. For every $f \in I$, $f \longrightarrow^+_{\mathcal{H}} 0$ if and only if there exist polynomials $a_1, \ldots, a_s \in K[x_1, \ldots, x_n]$ such that $f = a_1 f_1 + \cdots + a_s f_s$ and $deg(f) = \max_{1 \le i \le s} \{deg(a_i f_i)\}$.

Proof. Suppose that for every $f \in I$ there exist polynomials $a_1, \ldots, a_s \in K[x_1, \ldots, x_n]$ such that $f = a_1 f_1 + \cdots + a_s f_s$ and $deg(f) = \max_{1 \le i \le s} \{deg(a_i f_i)\}$. Then

$$H(f) = \sum_{deg(f) = deg(a_i f_i)} H(a_i) H(f_i).$$

$H(I) = \langle H(f_1), \ldots, H(f_s) \rangle$ means that $\mathcal{H} = \{f_1, f_2, \ldots, f_s\}$ is an H-basis. Hence $f \longrightarrow^+_{\mathcal{H}} 0$ by above lemma.

Suppose that $f \in I$ and $f \longrightarrow^+_{\mathcal{H}} 0$. Then

$$f = g_0 \rightarrow_{\mathcal{H}} g_1 \rightarrow_{\mathcal{H}} g_2 \rightarrow_{\mathcal{H}} \cdots \rightarrow_{\mathcal{H}} g_t = 0$$

for some polynomials $g_1, \ldots, g_t \in K[x_1, \ldots, x_n]$. If

$$H(g_{i-1}) = \sum_{j=1}^{s} a_{ij} H(f_j)$$

for $i = 1, \ldots, t$, then

$$f = \sum_{j=1}^{s} \sum_{i=1}^{t} a_{ij} f_j.$$

Q.E.D.

Using above lemmas, we can rewrite Theorem 2.5 with the new notation.

Theorem 4.3. Let $I = \langle h_1, h_2, \dots, h_t\rangle \subseteq K[x_1, \dots, x_n]$. Let the columns of the $t \times l$ matrix $S = (s_{ij})$ be a generating set of $syz(H(h_1), \dots, H(h_t))$. We may assume further that each $s_{ji}f_j$ is a homogeneous polynomial of same degree for $j = 1, \dots, t$. Then $\mathcal{H} = \{h_1, \dots, h_t\}$ is an H-basis for I if and only if

$$q_i = \sum_{j=1}^{t} s_{ji}h_j \longrightarrow_{\mathcal{H}}^{+} 0, \quad 1 \leq i \leq l.$$

Now we are ready to explain the new method for findings of the liftings of a given homogeneous ideal $J = \langle f_1, \dots, f_s\rangle \subset A$. First of all, we need to find a generating set for $\text{syz}(f_1, , f_s)$. Secondly, we define the polynomials

$$g_i = f_i + \sum_{\deg(x^\gamma)<\text{def}(f_i} C_{i\gamma}x^\gamma \quad 1 \leq i \leq s.$$

Then for each syzygy $(t_1, \dots, t_s)$ in the generating set of $\text{syz}(f_1, , f_s)$ define the polynomial $q_t = t_1g_1 + \cdots + t_sg_s$. Theorem 4.3 implies $(H) = \{g_1, \dots, g_s\}$ is an H-basis, in other words $I = \langle (H)\rangle$ is a lifting for J, if and only if every $q_t \longrightarrow_{\mathcal{H}}^{+} 0$. Therefore when applying the division algorithm, described in Lemma 3.3, to each q_t the conditions that the parameters $C_{i\gamma}$'s must satisfy for I to be a lifting of J can be obtained.

Let us illustrate the method with an example.

Example 4.4. Condider the ideal $J = \langle f_1, f_2, f_3\rangle = \langle x_1^2, x_1x_2, x_2^4 + x_1x_3^3\rangle$. Define the polynomials:

$g_1 = f_1 + C_1x_1 + C_2x_2 + C_3x_3 + C_4$,

$g_2 = f_2 + C_5x_1 + C_6x_2 + C_7x_4 + C_8$ and

$g_3 = f_3 + C_9x_1^3 + C_{10}x_1^2x_2 + C_{11}x_1^2x_3 + C_{12}x_1x_2^2 + C_{13}x_1x_2x_3 + C_{14}x_1x_3^2 + C_{15}x_2^3 + C_{16}x_2^2x_3 + C_{17}x_2x_3^2 + C_{18}x_3^3 + C_{19}x_1^2 + C_{20}x_1x_2 + C_{21}x_1x_3 + C_{22}x_2^2 + C_{23}x_2x_3 + C_{24}x_3^2 + C_{25}x_1 + C_{26}x_2 + C_{27}x_3 + C_{28}$.

The syzygy module $\text{syz}(f_1, f_2, f_3)$ can be generated $t_1 = (x_2, -x_1, 0), t_2 = (0, f_3, -f_2)$ and $t_3 = (x_3^3, x_2^3, -x_1)$. The details of computation of generators of syzygy modules can be found in [9]. Then we define polynomials:

$q_1 = x_2g_1 - x_1g_2 = -C_5x_1^2 + (C_1 - C_6)x_1x_2 + C_2x_2^2 - C_7x_1x_3 + C_3x_2x_3 - C_8x_1 + C_4x_2, q_2 = f_3g_2 - f_2g_3$ and $q_3 = x_3^3g_1 + x_23g_2 - x_1g_3$.

The ideal $I = \langle g_1, g_2, g_3\rangle$ is a lifting of J if and only if $q_1 \longrightarrow_{\mathcal{H}}^{+} 0$ for $i = 1, 2, 3$. Let us apply division algorithm to q_1.

$$H(q_1) = -C_5x_1^2 + (C_1 - C_6)x_1x_2 + C_2x_2^2 - C_7x_1x_3 + C_3x_2x_3.$$

Using the technique given in Example 3.1, $H(q_1) = -C_5f_1 + (C_1 - C_6)f_2$ under the condition $C_2 = C_3 = C_7 = 0$.

Let

$$\tilde{q_1} = q_1 + C_5g_1 - (C_1 - C_6)g_2 = (C_5C_6 - C_8)x_1 + (C_4 - C_1C_6 + C_6^2)x_2 + C_4C_5 - C_1C_8 + C_6C_8.$$

This a polynomial of degree 1, so it goes to the remainder. Since the remainder should be zero, the equations $C_8 = C_5C_6$ and $C_4 = C_1C_6 - C_6^2$ are obtained.

Applying same process to q_2 produce the following equations:

$C_6 = C_1 - C_{18}$,

$C_{24} = C_5C_{17} + C_{14}C_{18}$,

$C_{27} = -C_5^2C_{16} - C_5C_{13}C_{18} - C_{11}C_{18}^2 + C_{18}C_{21} + C_5C_{23}$ and

$C_{28} = -C_5^4 + C_5^3C_{15} + C_5^2C_{12}C_{18} + C_5C_{10}C_{18}^2 + C_9C_{18}^3 - C_{18}^2C_{19} - C_5C_{18}C_{20} - C_5^2C_{22} + C_{18}C_{25} + C_5C_{26}$

Replacing $C_6 = C_1 - C_{18}$ into equations $C_8 = C_5C_6$ and $C_4 = C_1C_6 - C_6^2$, we also get $C_4 = C_1C_{18} - C_{18}^2$ and $C_8 = C_1C_5 - C_5C_{18}$.

The division of q_3 does not produce additional equations. Hence $I = \langle g_1, g_2, g_3\rangle$ is a lifting of J if

$g_1 = x_1^2 + C_1x_1 + C_1C_{18} - C_{18}^2$,

$g_2 = x_1x_2 + C_5x_1 + (C_1 - C_{18})\, x_2 + C_1C_5 - C_5C_{18}$ and

$g_3 = x_2^4 + x_1x_3^3 + C_9x_1^3 + C_{11}x_1^2x_3 + C_{23}x_2x_3 + C_{13}x_1x_2x_3 + C_{16}x_2^2x_3 + (C_5C_{17} + C_{14}C_{18})\, x_3^2 + C_{14}x_1x_3^2 + C_{17}x_2x_3^2 + C_{18}x_3^3 + C_{19}x_1^2 + C_{10}x_1^2x_2 + C_{22}x_2^2 + C_{12}x_1x_2^2 + C_{15}x_2^3 + C_{25}x_1 + C_{26}x_2 + \left(-C_5^2C_{16} - C_5C_{13}C_{18} - C_{11}C_{18}^2 + C_{18}C_{21} + C_5C_{23}\right) x_3 - C_5^4 + C_5^3C_{15} + C_5^2C_{12}C_{18} + C_5C_{10}C_{18}^2 + C_9C_{18}^3 - C_{18}^2C_{19} - C_5C_{18}C_{20} - C_5^2C_{22} + C_{18}C_{25} + C_5C_{26}$.

Starting with a generating set for the homogeneous ideal J, the one need to add new polynomials to the generating set unless the original set is a Gröbner basis in the method suggested in [7]. This is the most important handicap of that method. In our method however we always use the given generating set of the ideal J. On the other hand, our method requires a generating set for the syzygy module. The best-known method for computation of a syzygy module uses Gröbner bases (see [9]). Even if the Gröbner basis of J is computed for finding a syzygy module, many S-polynomials do not produce a syzygy for J at the end. Hence the number of syzygies to be considered in our method is generally much less than the number of S-polynomials to be considered in the method given in[7]. Furthermore, we may only need a Gröbner basis for J not for I which contains polynomials with many parameters. The Gröbner basis computation with parameters might be very complicated.

References

[1] A. V. Geramita, D. Gregory and L. Roberts, *Monomial ideals and points in projective space*, J. Pure Appl. Algebra **40 (1)** (1986) 33–62.

[2] M. Roitman, *On the lifting problem for homogeneous ideals in polynomial rings*, J. Pure Appl. Algebra **51(1-2)** (1988) 205–215.

[3] Leslie G. Roberts, *On the lifting problem over an algebraically closed field*, C. R. Math. Rep. Acad. Sci. Canada **11(1)** (1989) 35–38.

[4] G. Carrá Ferro and , L. Robbiano, *On super G-bases*, J. Pure Appl. Algebra **68 (3)** (1990) 279–292.

[5] J. Migliore and U. Nagel, *Lifting monomial ideals*, Commun. Algebra **28(12)** (2000) 5679–5701.

[6] T. Luo and E. Yılmaz, *On the lifting problem for homogeneous ideals*, J. Pure Appl. Algebra **162(2-3)** (2001) 327–335.

[7] C. Bertone, F. Cioffi, M. Guida and M. Roggero , *The scheme of liftings and applications*, J. Pure Appl. Algebra **220** 2016 34–54.

[8] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties and Algorithms*, Springer (2015).

[9] D. Cox, J. Little and D. O'Shea, *Using Algebraic Geometry*, Springer (2005).